# Equimomental System of Rigidly Connected Equal Particles

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## Introduction

TWO rigid systems are said to be equimomental if their dynamic behaviors are identical. More specifically, equimomental dynamic systems possess the same total mass, location of center of mass, principal directions, and mass moments of inertia. For a three-dimensional rigid body, it is necessary to have at least four particles to represent its dynamic behavior. However, for a two-dimensional rigid body only three particles are needed in the equimomental system.

The study of the equimomental system is not new. Routh investigated the problem of equimomental system of particles nearly a century ago.[1] He concluded that the equimomental particles must lie on the surface of an equimomental ellipsoid which is centered at the center of mass of the rigid body. Routh pointed out that the equimomental system formed a tetrahedron with four equal particles. Each particle has a mass equal to one-fourth of the total mass and is placed at the vertex of the tetrahedron. If the equimomental ellipsoid is mapped onto a sphere by changing scales in the directions of coordinate axes, then the tetrahedron would be mapped onto a regular tetrahedron inscribed in the sphere. As the result of spherical symmetry, the orientation of the image tetrahedron remains arbitrary. Thus, the location of equimomental particles is not unique. The number of degrees of freedom is three in the three-dimensional problem and is one in the two-dimensional problem. Routh gave some examples of determination of the position of a particular set of equimomental particles through some simple mathematical manipulations. However, no procedure is presented for locating the general position of equimomental particles. Other particular solutions of equimomental systems are also given by Whittaker[2] and Greenwood.[3]

This Note presents a general procedure for locating the equimomental particles. Our solution will be expressed in terms of orientation parameters corresponding to the degree of freedom of the problem.

## Formulation of the Three-Dimensional Problem

Let us consider a three-dimensional rigid body of total mass $m$ and principal mass moments of inertia $I_x$, $I_y$, and $I_z$. We set the origin at the center of mass of the body and the $x$, $y$, and $z$ axes in the principal directions. Our objective is to locate four particles $P$, $Q$, $R$, and $S$ of mass $m/4$ at $(x_i, y_i, z_i)$ for $i = 1, 2, 3, 4$ such that the four mass system and a given rigid body are equimomental. Here we have

$$\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} y_i = \sum_{i=1}^{4} z_i = 0 \tag{1}$$

$$\sum_{i=1}^{4} x_i y_i = \sum_{i=1}^{4} y_i z_i = \sum_{i=1}^{4} z_i x_i = 0 \tag{2}$$

$$\sum_{i=1}^{4} x_i^2 = 4a^2/3, \qquad \sum_{i=1}^{4} y_i^2 = 4b^2/3, \qquad \sum_{i=1}^{4} z_i^2 = 4c^2/3 \tag{3}$$

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where

$$a^2 = 3/(2m)(-I_x + I_y + I_z) \tag{4}$$

$$b^2 = 3/(2m)(I_x - I_y + I_z) \tag{5}$$

$$c^2 = 3/(2m)(I_x + I_y - I_z) \tag{6}$$

Equations (1–3) include nine equations for 12 unknowns $x_i$, $y_i$, and $z_i$ $(i = 1, \ldots, 4)$. Hence the degree of freedom is equal to 3.

Assume that the position of the particle $P$ is known to be $(x_1, y_1, z_1)$. We may define

$$\bar{x}_i = x_i + x_1/3, \qquad \bar{y}_i = y_i + y_1/3, \qquad \bar{z}_i = z_i + z_1/3 \tag{7}$$

for $i = 2, 3, 4$, such that Eqs. (1–3) are reduced to

$$\sum_{i=2}^{4} \bar{x}_i = \sum_{i=2}^{4} \bar{y}_i = \sum_{i=2}^{4} \bar{z}_i = 0 \tag{8}$$

$$\sum_{i=2}^{4} \bar{x}_i\bar{y}_i = -\frac{4}{3}x_1y_1, \qquad \sum_{i=2}^{4} \bar{y}_i\bar{z}_i = -\frac{4}{3}y_1z_1$$
$$\sum_{i=2}^{4} \bar{z}_i\bar{x}_i = -\frac{4}{3}z_1x_1 \tag{9}$$

$$\sum_{i=2}^{4} \bar{x}_i^2 = \frac{4}{3}(a^2 - x_1^2), \qquad \sum_{i=2}^{4} \bar{y}_i^2 = \frac{4}{3}(b^2 - y_1^2)$$
$$\sum_{i=2}^{4} \bar{z}_i^2 = \frac{4}{3}(c^2 - z_1^2) \tag{10}$$

It is seen that the quantities $\bar{x}^2$, $\bar{y}^2$, $\bar{z}^2$, $\bar{x}\bar{y}$, $\bar{y}\bar{z}$, and $\bar{z}\bar{x}$ are components of a Cartesian tensor of order two. Hence the right-hand sides of Eqs. (9) and (10) are components of a second-order tensor in the $(\bar{x}, \bar{y}, \bar{z})$ system. The right-hand side of Eq. (8) is invariant under transformation of coordinates. Equations (8–10) represent an equimomental system of three particles. Note that the system of three particles can only represent a two-dimensional equimomental system. In other words, if we try to formulate the problem in the three-dimensional space, then one of the eigenvalues of the inertia matrix would be zero. This implies that the determinant of the inertia matrix must vanish. Hence

$$\begin{vmatrix} a^2 - x_1^2 & -x_1y_1 & -z_1x_1 \\ -x_1y_1 & b^2 - y_1^2 & -y_1z_1 \\ -z_1x_1 & -y_1z_1 & c^2 - z_1^2 \end{vmatrix} = 0 \tag{11}$$

Equation (11) leads to the following condition for $x_1$, $y_1$, and $z_1$:

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \tag{12}$$

Thus the particle $P$ must lie on the equimomental ellipsoid as defined by Eq. (12). Since $P$ is arbitrarily chosen, we conclude that all four particles $P$, $Q$, $R$, and $S$ must lie on the equimomental ellipsoid of semi-axes $a$, $b$, and $c$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{13}$$

The equimomental ellipsoid was first found by Routh.[1] It can be shown that the normal to the equimomental ellipsoid at any particle $P$ is perpendicular to the plane containing particles $Q$, $R$, and $S$. Furthermore, the right position of particles $P$, $Q$, $R$, and $S$ will make the volume of the tetrahedron with $P$, $Q$, $R$, and $S$ at its vertices the largest. It is equal to $(8/27)\sqrt{3}\,abc$.

## Computational Procedures

Let us define a new coordinate system $x'$, $y'$, and $z'$ such that

$$x = ax', \qquad y = by', \qquad z = cz' \tag{14}$$

In the $(x', y', z')$ system, the equimomental ellipsoid becomes a unit sphere

$$x'^2 + y'^2 + z'^2 = 1 \tag{15}$$

The image points $P'$, $Q'$, $R'$, and $S'$ of particles lie on the surface of the unit sphere. Due to spherical symmetry, these image points are interchangeable in the $(x', y', z')$ system and the image equimomental system forms a regular tetrahedron whose orientation is arbitrary. Let the image point $P'$ be at $(x_P', y_P', z_P')$ in the $(x', y', z')$ system. We shall find the positions of the remaining image points. Note that Eq. (15) can be viewed as a constraint condition. To specify the position of $P'$, only two degrees of freedom are required.

Let us rotate the coordinate axes about the origin. The new coordinate axes are $x''$, $y''$, and $z''$ axes. The rotation is made in such a manner that the $z''$ axis passes through the image point $P'$. Hence the position of $P'$ in the $(x'', y'', z'')$ system is

$$X_P'' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

Using the property of a regular tetrahedron, it is easy to find the position of the image points of particles $Q$, $R$, and $S$ in the $(x'', y'', z'')$ system to be

$$X_Q'' = \begin{Bmatrix} (2/3)\sqrt{2}\cos[\omega + (2/3)\pi] \\ (2/3)\sqrt{2}\sin[\omega + (2/3)\pi] \\ -1/3 \end{Bmatrix} \tag{17}$$

$$X_R'' = \begin{Bmatrix} (2/3)\sqrt{2}\cos\omega \\ (2/3)\sqrt{2}\sin\omega \\ -1/3 \end{Bmatrix} \tag{18}$$

$$X_S'' = \begin{Bmatrix} (2/3)\sqrt{2}\cos[\omega - (2/3)\pi] \\ (2/3)\sqrt{2}\sin[\omega - (2/3)\pi] \\ -1/3 \end{Bmatrix} \tag{19}$$

where $\omega$ is an arbitrary angle of rotation about the $z''$ axis. It can be identified as the third degree of freedom.

The coordinates $X'$ and $X''$ for any particle are related by the transformation relation

$$X' = CX'' \tag{20}$$

where $C$ is an orthogonal transformation matrix whose elements can be expressed in terms of Euler angles of rotation, $\phi$, $\theta$, and $\Psi$ from the $x', y', z'$ axes to the $x'', y'', z''$ axes. There are different versions in the definition of Euler angles in dynamics. In this study, our rotations of coordinate axes follow the order of $z$, $x$, $z$ axes, i.e., the third Euler angle $\Psi$ is the angle of rotation about the $z''$ axis. Since in Eqs. (17–19) the angle of rotation $\omega$ about the $z''$ axis is arbitrary, we may set $\Psi = 0$ and obtain

$$C = \begin{bmatrix} \cos\phi & -\cos\theta\sin\phi & \sin\theta\sin\phi \\ \sin\phi & \cos\theta\cos\phi & -\sin\theta\cos\phi \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{21}$$

We may use $\phi$, $\theta$, and $\omega$ as our orientation parameters. However, if we use $x_P$, $y_P$, and $\omega$ as the parameters of degree of freedom, we must apply Eq. (20) to the image point $P'$ and find

$$\begin{bmatrix} x_P' \\ y_P' \\ z_P' \end{bmatrix} = C \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

which leads to the following equations:

$$\sin\theta\sin\phi = x_P', \quad -\sin\theta\cos\phi = y_P', \quad \cos\theta = z_P' \tag{23}$$

Thus we can express $\phi$ and $\theta$ in terms of $x_P'$, $y_P'$, and $z_P'$ as

$$\phi = \tan^{-1}\left(-\frac{y_P'}{x_P'}\right), \qquad \theta = \cos^{-1} z_P' \tag{24}$$

The transformation matrix can then be expressed in terms of $x_P'$, $y_P'$, and $z_P'$ as

$$C = \begin{bmatrix} -\dfrac{y_P'}{D} & -\dfrac{x_P' z_P'}{D} & x_P' \\ \dfrac{x_P'}{D} & -\dfrac{y_P' z_P'}{D} & y_P' \\ 0 & D & z_P' \end{bmatrix} \tag{25}$$

where $D = (x_P'^2 + y_P'^2)^{1/2}$. After $C$ is determined, $X'$ and $X''$ for each particle can be calculated by Eqs. (16–24) and $X$ for each particle can be determined by Eq. (14).

## Example

In this example, we use $(x_P, y_P, \omega)$ as orientation parameters. We set $x_P = a$, $y_P = 0$, and $\omega = \pi/2$. It is found from Eq. (13) that $z_P = 0$ and

$$X_P = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

By Eqs. (14) and (25), we obtain

$$X_P' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By Eqs. (18–21) and Eq. (15), it is found that

$$X_Q' = \begin{bmatrix} -1/3 \\ -\sqrt{2/3} \\ -\sqrt{2}/3 \end{bmatrix}, \quad X_R' = \begin{bmatrix} -1/3 \\ 0 \\ (2/3)\sqrt{2} \end{bmatrix}, \quad X_S' = \begin{bmatrix} -1/3 \\ \sqrt{2/3} \\ -\sqrt{2}/3 \end{bmatrix}$$

and

$$X_Q = \begin{bmatrix} -1/3\,a \\ -\sqrt{2/3}\,b \\ -\sqrt{2}/3\,c \end{bmatrix}, \quad X_R = \begin{bmatrix} -1/3\,a \\ 0 \\ (2/3)\sqrt{2}\,c \end{bmatrix}, \quad X_S = \begin{bmatrix} -1/3\,a \\ \sqrt{2/3}\,b \\ -\sqrt{2}/3\,c \end{bmatrix}$$

which agree with the results given by Greenwood.[3]

## Conclusion

This Note provides a method for determination of location of each particle in an equimomental equal particle system of a given rigid body. Since the equimomental system has the same location of center of mass and the same principal moments of inertia, it posesses the same dynamic properties as the original rigid body. Routh[4] studied the mutual gravitational potential between two rigid bodies and obtained the first approximate expression for the gravitational potential for the case in which the distance between centers of mass of bodies is much larger than the size of each body. Only moments of inertia of each body are involved in the correction term of the gravitational potential. Thus, we conclude that the equimomental systems also have the same gravitational potential as the original rigid bodies up to the first approximation.

## References

[1]Routh, E. J., *Treatise on the Dynamics of a System of Rigid Bodies, Elementary Part,* Dover, New York, 1950, pp. 15–29.

[2]Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* Cambridge Univ. Press, Cambridge, England, UK, 1959, pp. 117–130.

[3]Greenwood, D. T., *Principles of Dynamics,* Prentice-Hall, Englewood Cliffs, NJ, 1965, problem 7.3 on p. 354 with solution on p. 506.

[4]Routh, E. J., *Treatise on the Dynamics of a System of Rigid Bodies, Advanced Part,* Dover, New York, 1955, pp. 340–346.

# Technical Comments

## Comment on "Generalized Technique for Inverse Simulation Applied to Aircraft Maneuvers"

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### Introduction

HESS et al.[1] introduced a new method for inverse simulation in their paper entitled "Generalized Technique for Inverse Simulation Applied to Aircraft Maneuvers"; this method is called the integration inverse method. The method assumes that the input is constant in the discretization interval $T$. The algorithm starts with an initial guess of the input. A forward simulation over $T$ follows. The variables at the end of the interval are then compared with the desired trajectory. Newton's method is then used to correct the initial guess of the input based on the Jacobian and the errors. The iterative procedure processes until the input converges. The input time history resulting from this method will be a step function in every interval $T$.

### Analysis

Consider the linear vehicle model:

$$\dot{x} = Ax + Bu \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

Let $nx$ be the number of states, $nu$ the number of inputs, and $ny$ the number of outputs. The authors of the paper[1] claimed that the method is independent of the values of $nx$, $nu$, and $ny$, provided $ny \le nu$. As the following example will show, in the situation where $nx > nu$ and $nx > ny$, the method may be unstable for small $T$.

Consider the second-order linear system:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = u(t) \tag{3}$$

Assume that $x(t)$ is specified by the desired trajectory $x_d(t)$, and $\dot{x}(t)$ is not specified. This represents a case that $nx = 2$ and $ny = nu = 1$. If the input $u(t) = \hat{u} =$ const over the period $t_0 \le t \le t_0 + T$, the solution of Eq. (3) is

$$x(t) = x_0 e^{-\sigma(t-t_0)}\left\{\cos[\omega_d(t-t_0)] + \frac{\sigma}{\omega_d}\sin[\omega_d(t-t_0)]\right\} + \frac{\dot{x}_0}{\omega_d}e^{-\sigma(t-t_0)}\sin[\omega_d(t-t_0)] + \frac{\hat{u}}{\omega_n^2}\left(1 - e^{-\sigma(t-t_0)}\left\{\cos[\omega_d(t-t_0)] - \frac{\sigma}{\omega_d}\sin[\omega_d(t-t_0)]\right\}\right) \tag{4}$$

where $\sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $x_0 = x(t_0)$, and $\dot{x}_0 = \dot{x}(t_0)$. The input in the interval $t_0 \le t \le t_0 + T$ can be solved as

$$\hat{u} = \omega_n^2\{x_1 - x_0 e^{-\sigma T}[\cos(\omega_d T) + (\sigma/\omega_d)\sin(\omega_d T)] - (\dot{x}_0/\omega_d)e^{-\sigma T}\sin(\omega_d T)\}/\{1 - e^{-\sigma T}[\cos(\omega_d T) - (\sigma/\omega_d)\sin(\omega_d T)]\} \tag{5}$$

where $x_1 = x(t_0 + T)$. If there are errors in the variables, the error of the input is

$$\Delta\hat{u} = \frac{\partial\hat{u}}{\partial x_1}\Delta x_1 + \frac{\partial\hat{u}}{\partial x_0}\Delta x_0 + \frac{\partial\hat{u}}{\partial\dot{x}_0}\Delta\dot{x}_0 \tag{6}$$

For the case that $T \ll 1$, the partial derivatives are $\partial\hat{u}/\partial x_1 \approx 2/T^2$, $\partial\hat{u}/\partial x_0 \approx -2/T^2$, and $\partial\hat{u}/\partial\dot{x}_0 \approx -2/T$.

In the ideal case, $x_1 = x_d(t_0 + T)$. However, because there are always some errors in the Newton's iteration and in the forward simulation, the term $\Delta x_1$ exists. $\Delta x_0 = 0$ if $x(t_0)$ is reset to $x_d(t_0)$ before the iteration procedure starts. The term $\Delta\dot{x}_0 \ne 0$ since the state variable $\dot{x}(t)$ is not specified. In summary, the error $\Delta x_1$ in each step is amplified by a factor $2/T^2$ then carried over to the next step through the variable $\dot{x}(t)$. The variable $\dot{x}(t)$ will drift away from the true value as time progresses. For very small $T$, the error grows quickly.

### Example

The solution of the differential equation

$$\ddot{x}(t) + \dot{x}(t) + x(t) = \sin 2t \tag{7}$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ is

$$x(t) = 1/13\ \{e^{-0.5t}[2\cos\sqrt{0.75}t + (7/\sqrt{0.75})\sin\sqrt{0.75}t] - 2\cos 2t - 3\sin 2t\} \tag{8}$$

The inverse simulation is formulated as follows: use Eq. (8) as the desired trajectory and find the input function.

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